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DYNAMIC STABILITY OF FLUID CONVEYING CANTILEVERED PIPES ON ELASTIC FOUNDATIONS

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1. INTRODUCTION

In 1972, Smith and Herrmann [1] studied the dynamic stability of a straight cantilever on a Winkler foundation (elastic foundation of constant stiffness) compressed by a follower force at the free end. They found that the critical load of such a cantilever does not depend on the foundation stiffness, being the same as the critical load of a cantilever without foundation. Later Hauger and Vetter [2] considered a cantilever loaded in a similar way but lying on an elastic foundation with variable stiffness. They showed that unlike the case of a Winkler foundation, variable elastic foundations can either reduce or increase the critical load of such a beam, depending on the form and magnitude of the foundation stiffness. Thus, variable elastic foundations have stabilizing or destabilizing effects on a straight cantilever compressed by an axial follower force at the free end compared to the case of the same cantilever without a foundation.

The dynamic stability of pipes conveying fluid has been studied for a long time (see references [3, 4] and the references therein). It has been established that a straight cantilevered pipe is stable for flow velocities less than a certain value, called the critical flow velocity. If such a pipe conveys fluid with the critical flow velocity it loses stability by flutter [3, 4]. Becker *et al.* [5] and Lottati and Kornecki [6] found that the critical flow velocity of a pipe on a Winkler foundation is higher than the critical flow velocity of the same pipe without a foundation (Becker *et al.* [5] considered only cantilevered pipes while Lottati and Kornecki [6] studied cantilevered as well as clamped–pinned pipes). Thus, a Winkler foundation is proved to have a stabilizing effect on fluid conveying cantilevered pipes unlike the case of cantilevered beams axially compressed by follower forces.

In references [7, 8] a model of pipes on multiparameter elastic foundations is suggested including, in particular, the case of cantilevered pipes lying on elastic foundations of Winkler and Pasternak types but with variable stiffness. However, in references [7, 8] the analysis finishes with the derivation of the governing equations and associated boundary conditions only. To the best of our knowledge, other studies on dynamic stability of pipes on variable elastic foundations are not reported in the literature (see the recent reviews in references [3, 4]).

In the present paper, the dynamic stability of fluid conveying straight cantilevered pipes lying on variable elastic foundation is investigated. The aim is to clarify whether the critical flow velocity depends on the form and magnitude of the foundation stiffness. The choice of the elastic foundations to be considered here is based on a recent systematic survey [7], where over 50 models of foundations are reviewed.

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2. DIFFERENTIAL EQUATION AND BOUNDARY CONDITIONS

The small transverse vibration of a straight pipe conveying inviscid fluid and lying on a variable elastic foundation of the Winkler type is governed by the linear fourth order partial differential equation (see references [7, 8])

$$EI\frac{\partial^4 w}{\partial x^4} + MU^2\frac{\partial^2 w}{\partial x^2} + 2MU\frac{\partial^2 w}{\partial x \partial t} + (m+M)\frac{\partial^2 w}{\partial t^2} + c(x)w = 0. \tag{1}$$

Here, $w(x, t)$ denotes the transverse displacement of the pipe neutral axis, x—the co-ordinate along this axis, t—the time, E—Young's modulus of the pipe material, I—the second moment of area of the pipe cross-section, m and M—the masses per unit length of the pipe and the fluid, respectively, U—the flow velocity, and $c(x)$—the variable foundation stiffness.

The pipe under consideration is assumed to be of the cantilevered type, i.e., its end $x = 0$ is fixed while the other one, $x = L$, is free, L being the pipe length. Hence, the boundary conditions are

$$w = 0, \quad \frac{\partial w}{\partial x} = 0 \quad \text{at } x = 0, \qquad \frac{\partial^2 w}{\partial x^2} = 0, \quad \frac{\partial^3 w}{\partial x^3} = 0 \text{ at } x = L. \tag{2}$$

3. APPROXIMATE SOLUTION

In this analysis, the Galerkin method [9] is applied to approximate the solutions of the differential equation (1) with boundary conditions (2). For that purpose the following functions are used:

$$W_n(x) = A_n\left\{\sin\frac{\Omega_n x}{L} - \sinh\frac{\Omega_n x}{L} - \frac{(\sin\Omega_n + \sinh\Omega_n)}{(\cos\Omega_n + \cosh\Omega_n)}\left(\cos\frac{\Omega_n x}{L} - \cosh\frac{\Omega_n x}{L}\right)\right\}, \tag{3}$$

$$\cos\Omega_n\cosh\Omega_n + 1 = 0, \qquad \Omega_n < \Omega_{n+1} \quad (n = 1, 2, \dots), \tag{4}$$

the constants A_n being such that

$$(W_i, W_j) = \delta_{ij} \quad (i, j = 1, 2, \dots), \tag{5}$$

where δ_{ij} is the Kronecker delta symbol; herein after

$$(f, g) = \int_0^L f(x)g(x)\,\mathrm{d}x, \tag{6}$$

denotes the scalar product of two arbitrary functions $f(x)$ and $g(x)$. As is well known, (see reference [10]) $\{W_n(x)\}_{n=1}^{\infty}$ forms a complete orthonormal set in the space of smooth functions of x on $[0, L]$, which satisfy boundary conditions (2).

Using functions of the form (3) as trial ones, an N-term approximate solution of equation (1) with boundary conditions (2) is sought in the form

$$w_N(x, t) = \sum_{n=1}^{N} T_n(t)\, W_n(x). \tag{7}$$

Substituting expression (7) into the left-hand side of equation (1), one obtains the residual function

$$R_N(x, t) = \sum_{n=1}^{N} \{(m + M) W_n \ddot{T}_n + 2MUW'_n \dot{T}_n + [MU^2 W''_n + (c(x) + EI\Omega_n^4) W_n] T_n\}, \quad (8)$$

which does not vanish identically since $w_N(x, t)$ is not an exact solution of equation (1). Here, and in the sequel, dots denote derivatives with respect to t and primes denote derivatives with respect to x. According to the standard Galerkin procedure [9], the unknown functions $T_n(t)$ are to be determined from the equations

$$(R_N, W_i) = 0 \quad (i = 1, 2, \dots, N) \quad (9)$$

in order to minimize residual (8). On account of formulae (3) and (5), equations (9) are written in the form

$$\sum_{j=1}^{N} \{(m + M)\delta_{ij} \ddot{T}_j + 2MUa_{ij} \dot{T}_j + (MU^2 b_{ij} + c_{ij} + EI\Omega_j^4 \delta_{ij}) T_j\} = 0, \quad (10)$$

where $i = 1, 2, \dots, N$, and

$$a_{ij} = (W_i, W'_j), \qquad b_{ij} = (W_i, W''_i), \qquad c_{ij} = (c(x) W_i, W_j). \quad (11)$$

The general solution of equations (10) is expressed (see reference [11]) by means of the roots $\lambda_1, \lambda_2, \dots, \lambda_{2N}$ of the equation

$$\det(\chi_{ij}) = 0, \quad (12)$$

where (χ_{ij}) is the characteristic λ-matrix associated with equations (10) whose elements are given by

$$\chi_{ij} = (m + M)\delta_{ij}\lambda^2 + 2MUa_{ij}\lambda + MU^2 b_{ij} + c_{ij} + EI\Omega_j^4 \delta_{ij}. \quad (13)$$

Thus, given a straight cantilevered pipe with parameters L, m, M, EI and U lying on an elastic foundation with stiffness $c(x)$, its small transverse vibration is approximated by a function of form (7), where $W_n(x)$ are of form (3) and $T_n(t)$ are solutions of equations (10). The convergence of this approximation is guaranteed, since the trial functions form a complete set.

It is well known (cf. references [4, 11]) that the knowledge of the roots $\lambda_1, \lambda_2, \dots, \lambda_{2N}$ of equation (12) is sufficient for the stability analysis. If for certain parameters L, m, M, EI, U and function $c(x)$, equation (12) has a multiple root with a zero real part or a root with a positive real part, then the associated pipe is unstable, otherwise the pipe is stable.

4. NUMERICAL RESULTS

The foregoing procedure for the determination of approximate solutions is implemented numerically using Maple tools embedded in Scientific WorkPlace version 2·5. The first 10 solutions of equation (4), the coefficients A_n in expressions (3), and the values of a_{ij}, b_{ij} and c_{ij} in expressions (11) are computed using 20-digit precision, otherwise the trial functions $W_1, \dots, W_{10}$ meet boundary conditions (2) unsatisfactorily. For similar reasons, the roots of equation (12) are calculated using 120-digit precision in all computations. In each example presented below, the critical flow velocity is obtained as the lowest value of U at which equation (12) has a multiple root with a zero real part or a root with a positive real part. Following reference [12], the obtained critical flow velocities U_c are presented in terms of

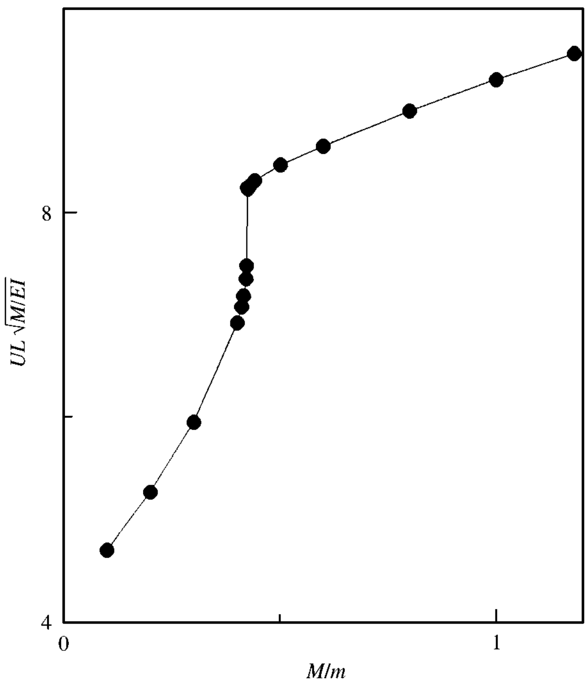

Figure 1. Our results (dots) compared to the results presented in reference [12] (solid line) for a cantilevered pipe without foundation.

dimensionless parameter $U_c L\sqrt{M/EI}$ and plotted versus the mass ratio M/m which is assumed to vary from 0·1 to 1·2. A typical 10 m long stainless-steel tube is considered with $EI = 3056937\ \mathrm{N\,m^2}$ and $m = 24{\cdot}498$ kg/m. The mass ratio $M/m = 1{\cdot}18$ indicates that the pipe under consideration conveys water.

First, the algorithm is tested by determining the critical flow velocities of a straight cantilevered pipe without foundation. It is found that the critical flow velocities obtained using 6, 7, 8 and 9 term approximations of the form (7) differ within 0·1%. An excellent agreement between these results and those presented in reference [12] can be observed in Figure 1. Therefore, all computations in the present study are carried out using six terms in approximation (7). Additional comments on the convergence of approximation (7) are given at the end of this section.

Next, elastic foundations with stiffness of the form

$$c(x) = \frac{EI}{L^4} k_0 \left\{ 4(1-\gamma)\left(\frac{x^2}{L^2} - \frac{x}{L}\right) + 1 \right\}, \tag{14}$$

suggested in reference [2] are considered, where k_0 and γ are given constants. The critical flow velocities are computed for three values of γ, namely $\gamma = 0, 1, 2$, and for each of them k_0 is taken to be 50, 100 and 150. For all values of γ and all mass ratios considered, the increments (in per cents) of the critical flow velocity corresponding to $k_0 = 150$ in comparison with the value obtained for $k_0 = 50$ have been computed.

The critical flow velocities in the case $\gamma = 1$, which corresponds to the Winkler foundation with stiffness $C_0 = k_0 EI/L^4$, are plotted in Figure 2. Inspecting these results one sees that the critical flow velocity of a straight cantilevered pipe on a Winkler foundation

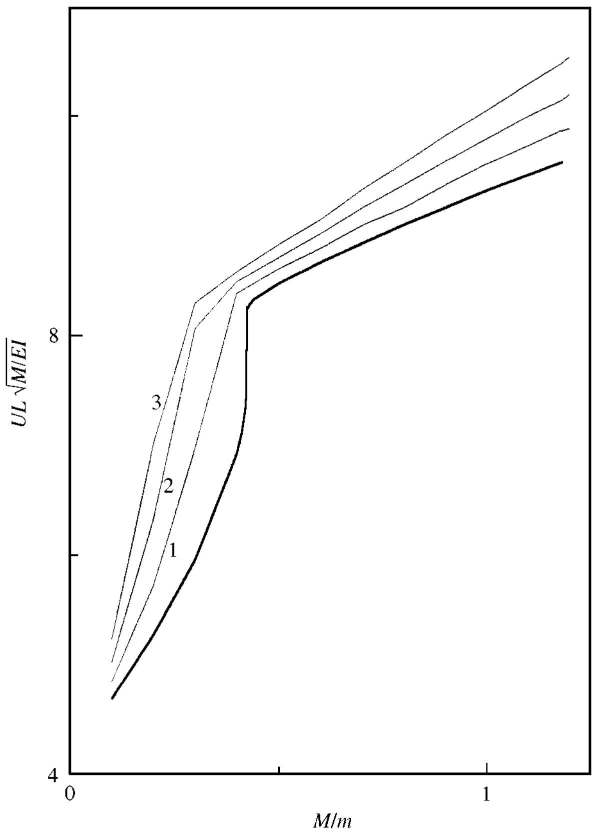

Figure 2. Critical flow velocity versus mass ratio of fluid conveying cantilevered pipes on Winkler foundations ($\gamma = 1$) with $k_0 = 50$ (curve 1), $k_0 = 100$ (curve 2) and $k_0 = 150$ (curve 3). The thick curve represents the critical velocity of a cantilevered pipe without foundation from reference [12].

increases together with the foundation stiffness C_0, which agrees with the results reported in references [5, 6]. The maximum increment of the critical flow velocity is achieved for $M/m = 0{\cdot}2$ and is approximately 22·5%.

The computed critical flow velocities for pipes on elastic foundations with stiffness of form (14) and $\gamma = 0$ (see Figure 3(a)) are presented in Figure 3(b). It is seen that the critical flow velocities in this case depend to a smaller degree on the foundation parameter k_0 in comparison with the case of the Winkler foundation. The maximum increment is about 14% and corresponds to $M/m = 0{\cdot}2$ again, but the majority of the other increments are within 1%.

The results for the critical flow velocities for $\gamma = 2$ (see Figure 4(a)) are shown in Figure 4(b). Among the three kinds of elastic foundations of stiffness (14) examined here, it provides the strongest dependence of the critical flow velocities on the parameter k_0. Indeed, the maximum increment is over 36%, but what is more important, the increments corresponding to high-density fluids are above 20%.

Finally, consider the dynamic stability of a straight cantilevered pipe lying on non-symmetric elastic foundations suggested in reference [13]. The stiffness of each of these foundations is assumed to be a polynomial of the form

$$c(x) = \frac{EI}{L^4}\left\{c_0 + \sum_{n=1}^{6} c_n\left(\frac{x}{L}\right)^n\right\}. \tag{15}$$

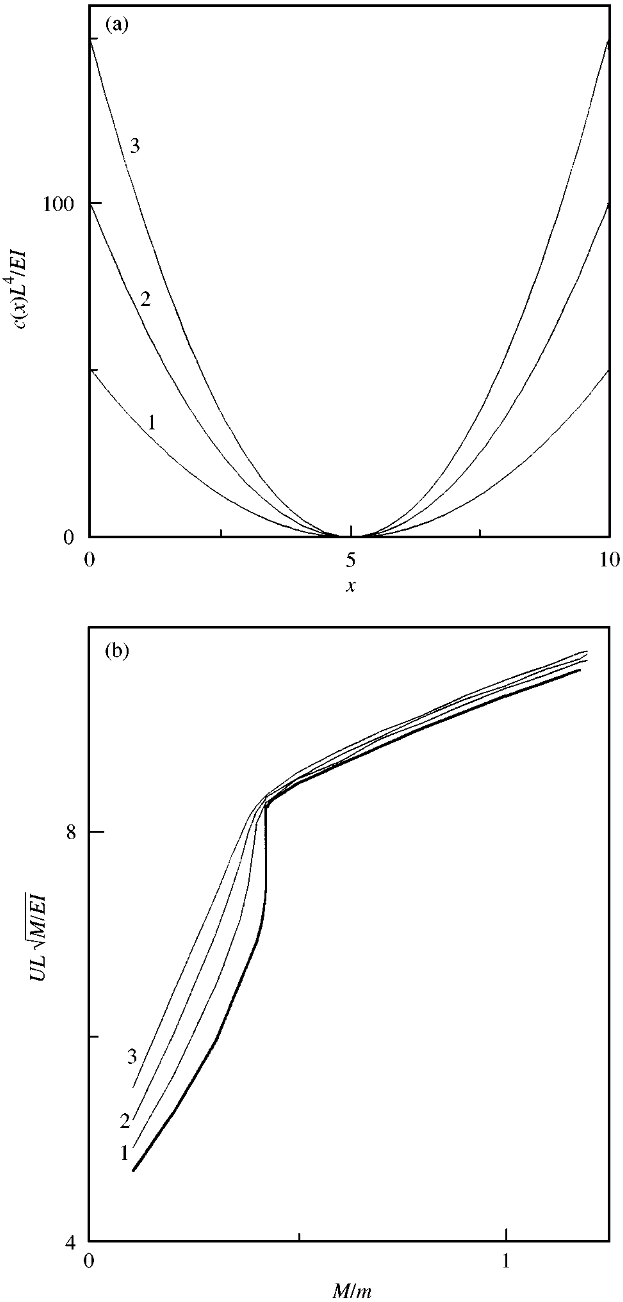

Figure 3. (a) Foundation stiffness of form (14) with $\gamma = 0$: $k_0 = 50$ (curve 1), $k_0 = 100$ (curve 2) and $k_0 = 150$ (curve 3). (b) Critical flow velocity of cantilevered pipes on such elastic foundations (the thicker curve represents the critical velocity of a cantilevered pipe without foundation from reference [12]).

Four of them are shown in Figure 5(a) and are used in the present study. The corresponding coefficients $c_0, \ldots, c_6$ are given in Table 1.

The critical flow velocities obtained are plotted in Figure 5(b). Obviously, strengthening of the foundation improves significantly (up to 82%) the stability of the pipe.

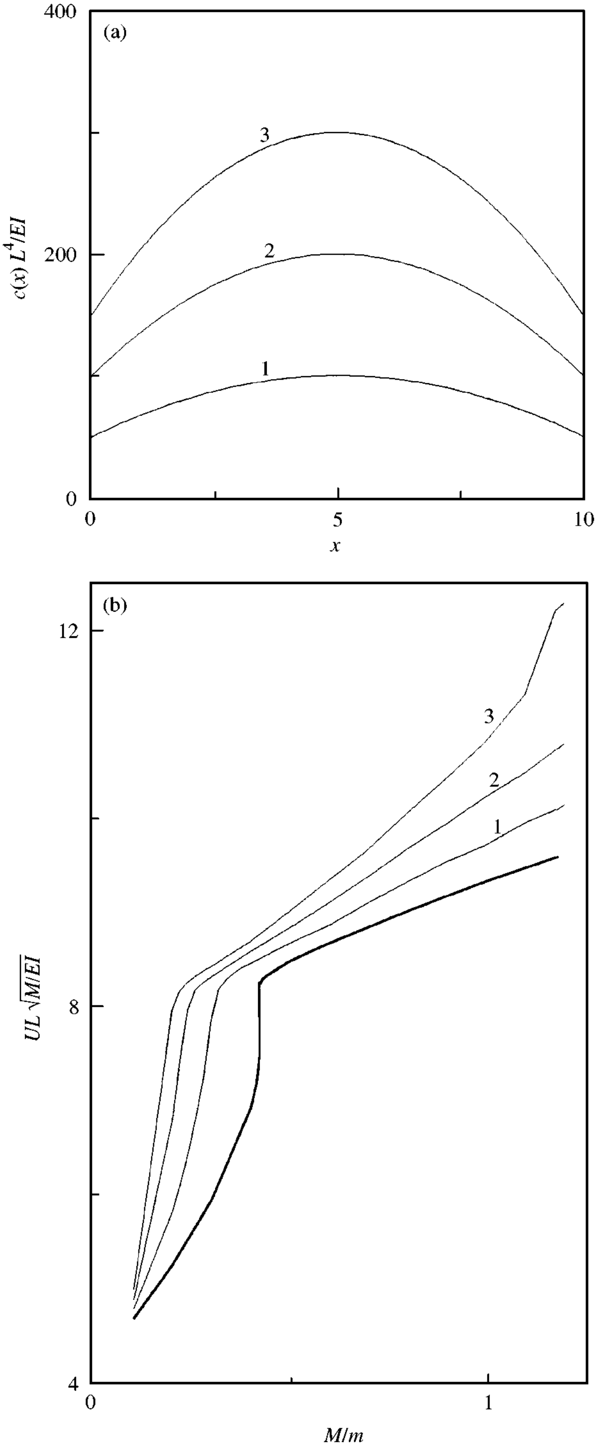

Figure 4. (a) Foundation stiffness of form (14) with $\gamma = 2$: $k_0 = 50$ (curve 1), $k_0 = 100$ (curve 2) and $k_0 = 150$ (curve 2). (b) Critical flow velocity of cantilevered pipes on such elastic foundations (the thicker curve represents the critical velocity of a cantilevered pipe without foundation from reference [12]).

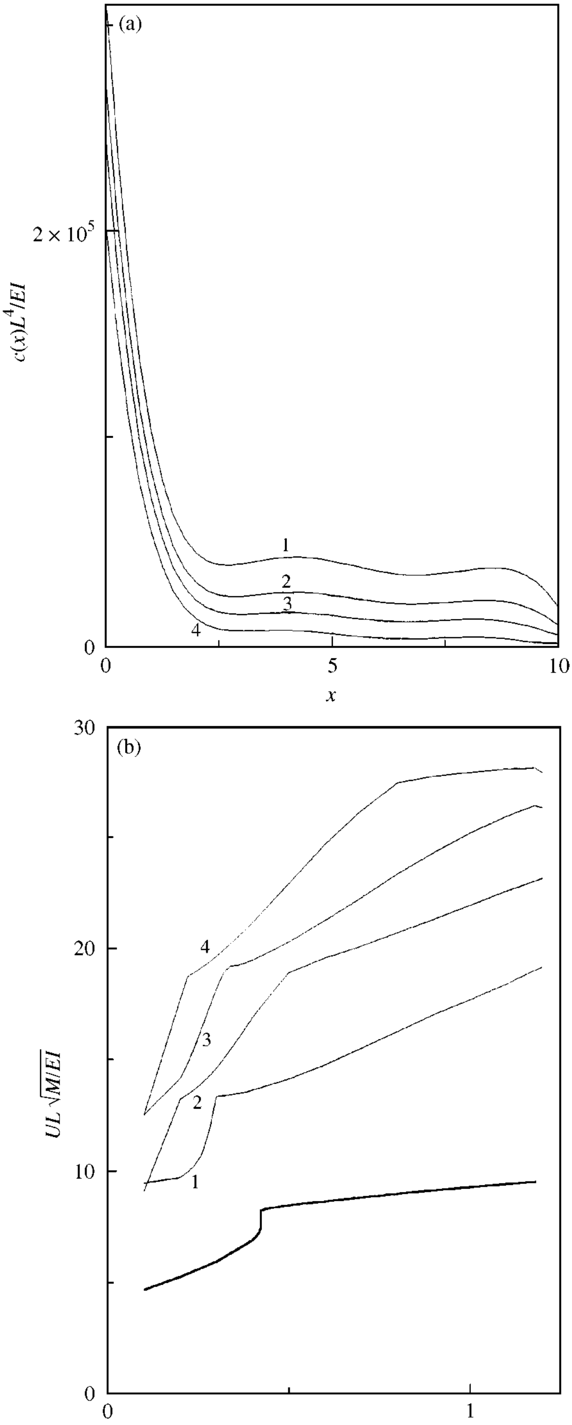

Figure 5. (a) Foundation stiffness of form (15): curves 1, 2, 3 and 4 are graphs of polynomial (15) with coefficients from Table 1, columns 1, 2, 3 and 4 respectively. (b) Critical flow velocity of cantilevered pipes on such elastic foundations (the thicker curve represents the critical velocity of a cantilevered pipe without foundation from reference [12]).

TABLE 1

Coefficients of the polynomials of form (15), *presented in Figure* 5(*a*)

	1	2	3	4
c_0	314746·9388	272755·4192	244578·856	203980·5991
c_1	− 338247·533	− 300995·478	− 278633·597	− 240905·249
c_2	159516·7264	141902·1647	133406·7593	116897·7352
c_3	− 36791·4077	− 33270·3676	− 32130·2919	− 28691·6920
c_4	4385·448358	4101·413111	4110·878005	3756·463305
c_5	− 256·745797	− 253·295714	− 266·252754	− 249·976918
c_6	5·746531432	6·143043014	6·853471674	6·641779563

In order to estimate the convergence rate of the numerical approach applied in this study, the critical flow velocities of pipes on certain elastic foundations obtained using 6, 7, 8 and 9 term approximations of form (7) are compared. The results of the present computations are compared with other results reported in the literature, where available.

First, pipes on Winkler foundations conveying water are analyzed. For the strongest Winkler foundation considered here, that is a foundation with stiffness of form (14) with $\gamma = 1$ and $k_0 = 150$, the difference between 6, 7, 8 and 9 term approximations is 0·29%.

On increasing the value of k_0, it was found that this difference is 0·3% for $k_0 = 10^3$ and 0·71% for $k_0 = 10^4$. The results of the computations for $k_0 = 10^3$ and 10^4 are also compared with the results in a recent study [13] and an excellent agreement is observed.

An elastic foundation of stiffness (14) with $k_0 = 150$ and $\gamma \neq 1$ was considered next. For $\gamma = 0$ it was found that the difference between the critical flow velocities obtained using 6, 7, 8 and 9 term approximations was 0·16% for $M/m = 0{\cdot}1$ and even less than 0·1% for $M/m = 0{\cdot}6$ and 1·18. For $\gamma = 2$ this difference is 0·18% for $M/m = 0{\cdot}1$, 0·23% for $M/m = 0{\cdot}6$, and negligible for $M/m = 1{\cdot}18$.

Finally, the most rigid elastic foundation whose stiffness is of form (15) (see curve 1 in Figure 5(a)) was considered. The difference between the critical flow velocities obtained using 6, 7, 8 and 9 term approximate solutions is 1·57% for $M/m = 0{\cdot}1$ and 1·5% for $M/m = 0{\cdot}6$. For mass ratio $M/m = 1{\cdot}18$, the maximum difference is 6·22% between 6 and 8 term approximate solutions but for 7, 8 and 9 term approximations, this difference is 0.89%.

5. CONCLUSIONS

Summarizing the above results, it can be seen that the critical flow velocity of a straight cantilevered pipe lying on a variable elastic foundation depends on both, the magnitude and the form of the foundation stiffness. It is established that the critical flow velocity of a pipe lying on any of the variable elastic foundations considered here is higher than the critical flow velocity of the same pipe without foundation. Hence, each foundation considered in this paper has a stabilizing effect on the straight cantilevered pipes.

In order to study the effect of the form of a symmetric elastic foundation on the critical flow velocity, the cases of foundation stiffness of form (14) suggested in reference [2] with $\gamma = 0$ and $\gamma = 2$ was considered. The above analysis shows that elastic foundations which are weak at the midpsan and have more strength at the ends of the pipe ($\gamma = 0$), have a less stabilizing effect even in comparison with Winkler foundations, whilst the elastic foundations that have more strength at the midspan and are weak at the ends ($\gamma = 2$) improve considerably the dynamic stability of a fluid conveying pipe.

Finally, considering the non-symmetric elastic foundations with stiffness of the form (15) plotted in Figure 5(a), it was found that strengthening these foundations also improves the dynamic stability of pipes conveying fluid.

It should be remarked that the above results are presented in dimensionless form, valid for arbitrary straight cantilevered pipes on elastic foundations of the types considered.

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